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Fixed Point for Chatterjea Contraction and Its Application to Cauchy Problem

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ABSTRACT

As an extension of the main result of Rathee et al. (2022), we establish a fixed point theorem in the framework of convex -metric spaces for Chatterjea contraction. Also, the fixed point is approximated by Kransnoselkij iterative procedure. We finally employ these findings to solve Cauchy problem.

Keywords: Metric Spaces, Convex Metric Spaces, Approximation, Chatterjea Contraction

INTRODUCTION AND PRELIMINARIES

The most useful and widely applied fixed point theorem in the field of fixed point theory ensuring the existence of fixed point for any contraction defined on complete metric spaces was proved by Stefan Banach[5] in 1922. In the literature, this result is also known with the name of Banach contraction principle. By introducing the notion of Chatterjea contraction in 1972, a generalization was set out of this principle by Chatterjea [6]. In 1989, the notion of b-metric spaces was introduced by Bakhtin[4] in lieu to extend this contraction principle. For more detail about this space one can refer to [8],[3], [1], [10], [2]. Chen et al. [7] introduced the concept of convex b-metric spaces in the recent years by using the concept of convex structure defined by Takahashi[13] in 1970.

Definition 1: *"Let $\mho \neq \phi$ and $s \geq 1$ (a real number). A mapping $\underline{\wp}_b: \mho \times \mho \to [0, \infty)$ is said to be a b-metric if the following holds for every $\underline{\varrho}, \underline{\varsigma}, \underline{\eta} \in \mho$*

1. $\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma}) = 0 \; iff \; \underline{\varrho} = \underline{\varsigma}$
2. $\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma}) = \underline{\wp}_b(\underline{\varsigma}, \underline{\varrho})$
3. $\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma}) \leq s[\underline{\wp}_b(\underline{\varrho}, \underline{\eta}) + \underline{\wp}_b(\underline{\eta}, \underline{\varsigma})]$

Further, a function $\underline{\Omega}: \mho \times \mho \times I \to \mho$ (where I = [0,1]) is said to have convex structure on $\mho$ if $\underline{\wp}_b(\underline{\eta}, \underline{\Omega}(\underline{\varrho}, \underline{\varsigma}; \vartheta)) \leq \vartheta \underline{\wp}_b(\underline{\eta}, \underline{\varrho}) + (1 - \vartheta)\underline{\wp}_b(\underline{\eta}, \underline{\varsigma})$ for each $\underline{\eta}, \underline{\varrho}, \underline{\varsigma} \in \mho$.

The triplet $(\mho, \underline{\wp}_b, \underline{\Omega})$ is called a convex b-metric space.

Additionally, by using the Mann's iterative algorithm, the authors established the Banach contraction principle in the framework of this lately introduced space, viz. convex b-metric space. In 2022, Rathee et al. [11] established a fixed point theorem for Chatterjea contraction and extended this result:

Theorem 1: *"Suppose $\underline{I}: (\mho, \underline{\wp}_b, \underline{\Omega}) \to (\mho, \underline{\wp}_b, \underline{\Omega})$ is a quasi-contraction, that is, $\underline{I}$ satisfies*

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) \leq \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})] \quad ...(1)$$

for all $\underline{\varrho}, \underline{\varsigma} \in \mho$ and some $\underline{\kappa} \in [0, \frac{1}{2})$, where $(\mho, \underline{\wp}_b, \underline{\Omega})$ is a complete convex b-metric space with $s > 1$. Let

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$\underline{\varrho}_n = \underline{\Omega}(\underline{\varrho}_{n-1}, \underline{I}\underline{\varrho}_{n-1}; \vartheta_{n-1})$ be a sequence defined by choosing an initial point $\underline{\varrho}_0 \in \mho$ with the property $\underline{\wp}_b\left(\underline{\varrho}_0, \underline{I}\underline{\varrho}_0\right) < \infty$, where $0 \le \vartheta_{n-1} < 1$ for each $n \in N$. If $\underline{\kappa} \le \frac{1}{s^2(s^2+1)}$ and $\vartheta_{n-1} < \frac{\frac{1}{s^4} - \frac{\underline{\kappa}}{s^2} - \underline{\kappa}}{\frac{1+\underline{\kappa}}{s} - \underline{\kappa}}$ for each $n \in N$, then $\underline{I}$ has a fixed point in $\mho$ that is unique.

In the present work, we improve this theorem by enlarging the range of $\underline{\kappa} \in [0, \frac{1}{2})$. Furthermore, the fixed point is computed by means of Kransnoselkij iteration. Moreover, some examples are presented to prove the universality of the proven results over Theorem 1 as well as over the similar results existing in the literature. As an application, we arrive at the solution for the Cauchy problem.

MAIN RESULT

We start this section with the following lemma that is required in the sequel to assure the existence and approximation of fixed point.

Lemma 1: Let $\underline{I}: \mho \to \mho$ be a self mapping defined on $(\mho, \underline{\wp}_b)$, a complete b-metric space, such that for all $\underline{\varrho}, \underline{\varsigma} \in \mho$ and $\underline{\kappa} \in [0, \frac{1}{2})$, it satisfies

$$\underline{\wp}_b(\underline{I}\underline{\varrho}, \underline{I}\underline{\varsigma}) \le \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I}\underline{\varrho})] \qquad \dots(2)$$

If $\underline{\kappa} < \frac{1}{s^2}$, then $\underline{I}$ has a unique fixed point if and only if $\underline{I}$ has approximate fixed point property, i.e. $inf\{\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho});\ \underline{\varrho} \in \mho\} = 0$.

Proof. Firstly, assume that a unique fixed point of $\underline{I}$, say $\underline{\varrho}$, exists, i.e., $\underline{I}\underline{\varrho} = \underline{\varrho}$. Then,

$\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) = 0,$

$inf\{\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho});\ \underline{\varrho} \in \mho\} = 0.$

Thus, $\underline{I}$ exhibits approximate fixed point property.

Conversely, assume that $\underline{I}$ exhibits approximate fixed point property, i.e. $inf\{\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho});\ \underline{\varrho} \in \mho\} = 0$. This indicates the existence of $\langle\underline{\varrho}_n\rangle_{n\in N}$, a sequence in $\mho$ satisfying $lim_{n\to\infty}\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n) = 0$ and by using (2) and triangle inequality for all $m, n \in \mho$,

$$\underline{\wp}_b(\underline{I}\underline{\varrho}_n, \underline{I}\underline{\varrho}_m) \le \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_m) + \underline{\wp}_b(\underline{\varrho}_m, \underline{I}\underline{\varrho}_n)]$$
$$\le \underline{\kappa}[s\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n) + 2s\underline{\wp}_b(\underline{I}\underline{\varrho}_n, \underline{I}\underline{\varrho}_m)$$
$$+s\underline{\wp}_b(\underline{\varrho}_m, \underline{I}\underline{\varrho}_m)]\ (1 - 2\underline{\kappa}s)\underline{\wp}_b(\underline{I}\underline{\varrho}_n, \underline{I}\underline{\varrho}_m) \le$$
$$\underline{\kappa}s[\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n)\ + \underline{\wp}_b(\underline{I}\underline{\varrho}_m, \underline{\varrho}_m)]$$

Now, since $lim_{n\to\infty}\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n) = 0$ and $\underline{\kappa} < \frac{1}{s^2}$, we obtain a Cauchy sequence $\langle\underline{I}\underline{\varrho}_n\rangle_{n\in N}$ as $n \to \infty$. Also, there exists an element $\underline{\varrho} \in \mho$ satisfying $lim_{n\to\infty}\underline{I}\underline{\varrho}_n = \underline{\varrho}$ as the space $(\mho, \underline{\wp}_b)$ is complete. Again using triangle inequality

$$\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}) \le s[\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n) + \underline{\wp}_b(\underline{I}\underline{\varrho}_n, \underline{\varrho})].$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty}\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}) = 0\underline{\varrho}_n \to \underline{\varrho}.$$

Also, consider

$$\underline{\wp}_b(\underline{I}\underline{\varrho}_n, \underline{I}\underline{\varrho}) \le \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}_n, \underline{I}\underline{\varrho}) + \underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}_n)].$$

Now taking limit as $n \to \infty$,

$$\frac{1}{s}\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) \le \underline{\kappa}[s\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) + 0]$$

$$= \underline{\kappa}s\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho})$$

$$\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) \le \underline{\kappa}s^2\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho})$$

$$(1 - \underline{\kappa}s^2)\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) \le 0$$

Since $\underline{\kappa} < \frac{1}{s^2}$, i.e., $1 - \underline{\kappa}s^2 < 1$, we get

$$\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varrho}) = 0\underline{I}\underline{\varrho} = \underline{\varrho}$$

and $\underline{\varrho} \in \mho$. Thus $\underline{I}$ has a fixed point in $\mho$.

If possible, now consider two fixed points of $\underline{I}$, say $\underline{\varrho}$ and $\underline{\varsigma}$, exist and thus $\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma}) \ne 0$. By using inequality (2), we get

$$\underline{\wp}_b\left(\underline{\varrho}, \underline{\varsigma}\right) = \underline{\wp}_b\left(\underline{I}\underline{\varrho}, \underline{I}\underline{\varsigma}\right)$$

$$\le\ \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}, \underline{I}\underline{\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I}\underline{\varrho})]$$

$$\le 2\underline{\kappa}\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma})$$

that is a contradiction as $\kappa \in [0,\frac{1}{2})$.

$\wp_b(\varrho,\varsigma) = 0$ and thus $\varrho = \varsigma$, i.e. the fixed point is unique.

Theorem 2: *Let $I:\mho \to \mho$ be a self mapping defined on $(\mho,\wp_b,\Omega)$, a complete convex b-metric space with parameter $s \geq 2$ such that for all $\varrho,\varsigma \in \mho$ and $\kappa \in [0,\frac{1}{2})$, it satisfies*

$$\wp_b\left(I\varrho,I\varsigma\right) \leq \kappa\left[\wp_b\left(\varrho,I\varsigma\right) + \wp_b\left(\varsigma,I\varrho\right)\right] \qquad \ldots(3)$$

Then, I has approximate fixed point property.

Proof. For every $\varrho \in \mho$, we have

$$\wp_b(I^{n+1}\varrho,I^n\varrho) \leq \kappa[\wp_b(I^n\varrho,I^n\varrho)$$
$$+\wp_b(I^{n-1}\varrho,I^{n+1}\varrho)]$$
$$\leq \kappa s[\wp_b(I^{n-1}\varrho,I^n\varrho) + \wp_b(I^n\varrho,I^{n+1}\varrho)]$$
$$(1-\kappa s)\wp_b(I^{n+1}\varrho,I^n\varrho) \leq \kappa s\wp_b(I^{n-1}\varrho,I^n\varrho)$$
$$\wp_b(I^{n+1}\varrho,I^n\varrho) \leq \frac{\kappa s}{1-\kappa s}\wp_b(I^{n-1}\varrho,I^n\varrho)$$
$$< \wp_b(I^{n-1}\varrho,I^n\varrho)$$

since $\kappa < \frac{1}{s^2}$. Thus the sequence $\langle\wp_b(I^{n+1}\varrho,I^n\varrho)\rangle_{n\in N}$ is non-increasing and for $\lambda \in N$,

$$\wp_b\left(I^{\lambda+1}\varrho,I^{\lambda}\varrho\right) < \wp_b\left(I^{\lambda}\varrho,I^{\lambda-1}\varrho\right)$$
$$< \cdots < \wp_b(I^2\varrho,I\varrho) < \wp_b(I\varrho,\varrho).$$

Now, consider

$$\wp_b\left(I^{\lambda}\varrho,I^{\lambda+2}\varrho\right) \leq \kappa\left[\wp_b\left(I^{\lambda-1}\varrho,I^{\lambda+2}\varrho\right) + \wp_b\left(I^{\lambda+1}\varrho,I^{\lambda}\varrho\right)\right]$$
$$< \kappa\left[\wp_b\left(\varrho,I\varrho\right) + \wp_b\left(I^{\lambda-1}\varrho,I^{\lambda+2}\varrho\right)\right]$$
$$\leq \kappa[\wp_b(\varrho,I\varrho) + s\{\wp_b(I^{\lambda-1}\varrho,I^{\lambda}\varrho)$$
$$+\wp_b(I^{\lambda}\varrho,I^{\lambda+2}\varrho)\}]$$
$$< \kappa(s+1)\wp_b(\varrho,I\varrho) + \kappa s\wp_b(I^{\lambda}\varrho,I^{\lambda+2}\varrho)$$

$$(1-\kappa s)\wp_b\left(I^{\lambda}\varrho,I^{\lambda+2}\varrho\right) \leq \kappa(s+1)\wp_b\left(\varrho,I\varrho\right)$$
$$\wp_b(I^{\lambda}\varrho,I^{\lambda+2}\varrho) \leq \frac{\kappa(s+1)}{1-\kappa s}\wp_b(\varrho,I\varrho)$$
$$= \varsigma\wp_b(\varrho,I\varrho) \qquad \ldots(4)$$

where $\varsigma = \frac{\kappa(s+1)}{1-\kappa s}$.

We let $inf\{\wp_b(\varrho,I\varrho);\ \varrho \in \mho\} = \Lambda$. We need to prove that this $\Lambda = 0$. For this, let $\langle\varrho_n\rangle$ be a sequence such that $lim_{n\to\infty}\wp_b(\varrho_n,I\varrho_n) = \Lambda$, i.e., by (4), we have, for every $n \in N$ and some $\lambda(n) \in N$,

$$\wp_b(I^{\lambda(n)}\varrho_n,I^{\lambda(n)+2}\varrho_n) \leq \varsigma\wp_b(\varrho_n,I\varrho_n). \qquad \ldots(5)$$

Now, $(\mho,\wp_b,\Omega)$ being a complete convex b-metric space, defining $\eta_n = \Omega(I^{\lambda(n)+1}\varrho_n,I^{\lambda(n)+2}\varrho_n,\vartheta)$ leads to a well defined η_n belonging to $\mho$, where we can choose $\vartheta \in (0,1)$ such that $\vartheta < \frac{\frac{1}{2}-s\kappa}{\kappa}$ and then we have,

$$\wp_b\left(\eta_n,I\eta_n\right) \leq \vartheta\wp_b\left(I^{\lambda(n)+1}\varrho_n,I\eta_n\right)$$
$$+ (1-\vartheta)\wp_b\left(I^{\lambda(n)+2}\varrho_n,I\eta_n\right)$$
$$\leq \kappa\vartheta[\wp_b(I^{\lambda(n)}\varrho_n,I\eta_n)$$
$$+ \kappa(1-\vartheta)[\wp_b(I^{\lambda(n)+1}\varrho_n,I\eta_n)$$
$$+ \wp_b(\eta_n,I^{\lambda(n)+1}\varrho_n)]$$
$$+ \wp_b(\eta_n,I^{\lambda(n)+2}\varrho_n)]$$
$$\leq \kappa\vartheta[s\wp_b(I^{\lambda(n)}\varrho_n,\eta_n) + s\wp_b(\eta_n,I\eta_n) +$$
$$+\wp_b(\eta_n,I^{\lambda(n)+1}\varrho_n)]$$
$$+\kappa(1-\vartheta)[s\wp_b(I^{\lambda(n)+1}\varrho_n,\eta_n)$$
$$+s\wp_b(\eta_n,I\eta_n) + \wp_b(\eta_n,I^{\lambda(n)+2}\varrho_n)]$$
$$\leq \kappa\vartheta[s\vartheta\wp_b(I^{\lambda(n)}\varrho_n,I^{\lambda(n)+1}\varrho_n) + s\wp_b(\eta_n,I\eta_n)$$
$$+s(1-\vartheta)\wp_b(I^{\lambda(n)}\varrho_n,I^{\lambda(n)+2}\varrho_n)$$
$$+(1-\vartheta)\wp_b(I^{\lambda(n)+2}\varrho_n,I^{\lambda(n)+1}\varrho_n)]$$
$$+\kappa(1-\vartheta)[s(1-\vartheta)\wp_b(I^{\lambda(n)+1}\varrho_n,I^{\lambda(n)+2}\varrho_n)$$

$$+s\wp_b(\underline{\eta_n},\underline{I}\underline{\eta_n})+\vartheta\wp_b(\underline{I}^{\lambda(n)+1}\underline{\varrho_n},\underline{I}^{\lambda(n)+2}\underline{\varrho_n})]$$

$$\leq \underline{\kappa}\vartheta[s\vartheta\wp_b(\underline{\varrho_n},\underline{I}\underline{\varrho_n})+s(1-\vartheta)\varsigma\wp_b(\underline{\varrho_n},\underline{I}\underline{\varrho_n})$$

$$+s\wp_b(\underline{\eta_n},\underline{I}\underline{\eta_n})+(1-\vartheta)\wp_b(\underline{\varrho_n},\underline{I}\underline{\varrho_n})]$$

$$+\underline{\kappa}(1-\vartheta)[s(1-\vartheta)\wp_b(\underline{\varrho_n},\underline{I}\underline{\varrho_n})$$

$$+s\wp_b(\underline{\eta_n},\underline{I}\underline{\eta_n})+\vartheta\wp_b(\underline{\varrho_n},\underline{I}\underline{\varrho_n})]. \quad \ldots(6)$$

Suppose $lim_{n\to\infty}\wp_b(\underline{\eta_n},\underline{I}\underline{\eta_n})=\underline{\delta}$.

We notice that $\underline{\delta}$ is finite, and also, $\underline{\Lambda}\leq\underline{\delta}$ is true. Next, we claim that $\underline{\delta}=0$ which shall prove that $\underline{\Lambda}=0$.

Taking $limsup$ as $n\to\infty$ on both sides of inequality (6) and using $\underline{\Lambda}\leq\underline{\delta}$

$$\underline{\delta}\leq\underline{\kappa}\vartheta[s\vartheta\underline{\delta}+s(1-\vartheta)\rho\underline{\delta}+s\underline{\delta}+(1-\vartheta)\underline{\delta}]$$

$$+\underline{\kappa}(1-\vartheta)[s(1-\vartheta)\underline{\delta}+s\underline{\delta}+\vartheta\underline{\delta}]$$

$$=\underline{\kappa}[2s\vartheta^2+s\vartheta(1-\vartheta)\rho+2\vartheta(1-\vartheta)$$

$$+2s(1-s)]\underline{\delta}$$

$$=\underline{\kappa}s[(\rho-2)\vartheta(1-\vartheta)+2]\underline{\delta}+\underline{\kappa}\vartheta(1-\vartheta)\underline{\delta}$$

$$\leq\underline{\kappa}[2s+2\vartheta(1-\vartheta)]\underline{\delta}.$$

If possible, suppose that $\underline{\delta}>0$. Then, by inequality (6), we get

$$1\leq\underline{\kappa}[2s+2\vartheta(1-\vartheta)]$$

$$\leq 2s\underline{\kappa}+2\underline{\kappa}\vartheta$$

$$\vartheta\geq\frac{\frac{1}{2}-s\underline{\kappa}}{\underline{\kappa}},$$

which is a contradiction since $\vartheta<\frac{\frac{1}{2}-s\underline{\kappa}}{\underline{\kappa}}$.

Thus, $\underline{I}$ has approximate fixed point property.

Lemma 2: *Let* $(\mho,\wp_b,\underline{\Omega})$ *be a convex b-metric space. Define self-mappings* $\underline{I}:\mho\to\mho$ *and* $\underline{I}_\vartheta:\mho\to\mho$ *by*

$$\underline{I}_\vartheta\underline{\varrho}=\underline{\Omega}(\underline{\varrho},\underline{I}\underline{\varrho};\vartheta),\underline{\varrho}\in\mho.$$

Then, for any $\vartheta\in[0,1)$,

$$Fix(\underline{I})=Fix(\underline{I}_\vartheta).$$

Proof. By definition,

$$\underline{I}_\vartheta=\vartheta\underline{\varrho}+(1-\vartheta)\underline{I}\underline{\varrho}.$$

If $\vartheta=0$, then

$\underline{I}_\vartheta\underline{\varrho}=\underline{I}\underline{\varrho}\ \forall\ \underline{\varrho}\in\mho$

i.e., $\underline{I}_\vartheta=\underline{I}$

$Fix(\underline{I})=Fix(\underline{I}_\vartheta)$.

Now assume that $\vartheta\in(0,1)$ and let a fixed point of $\underline{I}$, say $\underline{\varrho}^*$, exists, i.e., $\underline{\varrho}^*=\underline{I}\underline{\varrho}^*$ and therefore,

$\wp_b(\underline{\varrho}^*,\underline{I}_\vartheta\underline{\varrho}^*)=\wp_b(\underline{\varrho}^*,\underline{\Omega}(\underline{\varrho}^*,\underline{I}\underline{\varrho}^*;\vartheta))$

$\leq\vartheta\wp_b(\underline{\varrho}^*,\underline{\varrho}^*)+(1-\vartheta)\wp_b(\underline{\varrho}^*,\underline{I}\underline{\varrho}^*)=0$

$\underline{\varrho}^*=\underline{I}_\vartheta\underline{\varrho}^*$

i.e., $\underline{\varrho}^*$ is a fixed point of $\underline{I}_\vartheta$.

Conversely, presume that $\underline{\varrho}^*$ is a fixed point of $\underline{I}_\vartheta$, i.e., $\wp_b(\underline{\varrho}^*,\underline{I}_\vartheta\underline{\varrho}^*)=0$, then

$$\wp_b(\underline{\varrho}^*,\underline{\Omega}(\underline{\varrho}^*,\underline{I}\underline{\varrho}^*;\vartheta))=0$$

$$\vartheta\wp_b(\underline{\varrho}^*,\underline{\varrho}^*)+(1-\vartheta)\wp_b(\underline{\varrho}^*,\underline{I}\underline{\varrho}^*)=0^{(6)}$$

$$(1-\vartheta)\wp_b(\underline{\varrho}^*,\underline{I}\underline{\varrho}^*)=0.$$

Since $\vartheta\neq1$, $\underline{\varrho}^*$ is a fixed point of $\underline{I}$.

Hence, the proof.

The following result is an extension of Chatterjea fixed point theorem in the case of Convex b-metric spaces by Rathee et al. [11]

Theorem 3: *Let* $\underline{I}:\mho\to\mho$ *be a self mapping defined on* $(\mho,\wp_b,\underline{\Omega})$, *a complete convex b-metric space with parameter* $s\geq2$ *such that for all* $\underline{\varrho},\underline{\varsigma}\in\mho$ *and* $\underline{\kappa}\in[0,\frac{1}{2})$, *it satisfies*

$$\wp_b(\underline{I}\underline{\varrho},\underline{I}\underline{\varsigma})\leq\underline{\kappa}[\wp_b(\underline{\varrho},\underline{I}\underline{\varrho})+\wp_b(\underline{\varsigma},\underline{I}\underline{\varsigma})] \quad \ldots(7)$$

If $\underline{\kappa}<\frac{1}{s^2}$, then

1. A fixed point of $\underline{I}$, say $\underline{\varrho}$, exists that is unique.

2. The sequence $\langle \underline{\varrho}_n \rangle_{n\in N}$ converges to $\underline{\varrho}$ for any $\underline{\varrho}_0 \in \mho$ that is obtained from the iterative procedure

   $\underline{\varrho}_{n+1} = \underline{\Omega}(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n; \vartheta);\ n \geq 0.$

3. The error estimate

   $$\frac{1}{s}\underline{\wp}_b(\underline{\varrho}_{n+i-1}, \underline{\varrho}) \leq \frac{\underline{\xi}^i}{1-\underline{\xi}}\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n-1})$$

   holds for $n = 1,2,\cdots;\ i = 1,2,\cdots$.

Proof.

1. Lemma 1 and Theorem 2 concludes the proof.

2. We observe that $\underline{\varrho}_{n+1} = \underline{\Omega}(\underline{\varrho}_n, \underline{I}\underline{\varrho}_n; \vartheta)$, i.e.

$$\underline{\varrho}_{n+1} = \underline{I}_\vartheta \underline{\varrho}_n;\ n \geq 0.$$

Taking $\underline{\varrho} = \underline{\varrho}_n$ and $\underline{\varsigma} = \underline{\varrho}_{n-1}$, in (7)

$$\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n) \leq \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_n) + \underline{\wp}_b(\underline{\varrho}_{n-1}, \underline{\varrho}_{n+1})]$$

$$\leq \underline{\kappa}s[\underline{\wp}_b(\underline{\varrho}_{n-1}, \underline{\varrho}_n) + \underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n+1})]$$

implying

$$\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n) \leq \frac{\underline{\kappa}s}{1-\underline{\kappa}s}\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n-1})$$

$$\leq \underline{\kappa}s^2\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n-1})$$

$$= \underline{\xi}\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n-1}), say, \underline{\kappa}s^2 = \underline{\xi}$$

$$\vdots$$

$$\leq \underline{\xi}^n\underline{\wp}_b(\underline{\varrho}_1, \underline{\varrho}_0) \qquad \ldots(8)$$

As $\underline{\xi} \in [0,1)$, we have

$$\lim_{n\to\infty}\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n) = 0. \ (9)$$

Now consider the points $\underline{\varrho}$ and $\underline{\varsigma}$ as $\underline{\varrho}_{n+k}$ and $\underline{\varrho}_n$, respectively, in inequality (7).

$$\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1}) \leq \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+1})$$
$$+\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n+k+1})]$$
$$\leq \underline{\kappa}s[\underline{\wp}_b(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+k+1}$$
$$+ 2\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1})\underline{\wp}_b(\underline{\varrho}_n, \underline{\varrho}_{n+1})]$$

This implies

$$(1-2\underline{\kappa}s)\underline{\wp}_b\left(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1}\right) \leq \underline{\kappa}s\underline{\wp}_b\left(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+k+1}\right) + \underline{\kappa}s\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n)$$

$$(1-\underline{\kappa}s^2)\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1}) < \underline{\kappa}s^2\underline{\wp}_b\left(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+k+1}\right) + \underline{\kappa}s^2\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n)$$

$$(1-\underline{\xi})\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1}) \leq \underline{\xi}[\underline{\wp}_b(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+k+1}) + \underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n)]$$

and $\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+1})$

$$\leq \frac{\underline{\xi}}{1-\underline{\xi}}[\underline{\wp}_b(\underline{\varrho}_{n+k}, \underline{\varrho}_{n+k+1}) + \underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n)] \ (10)$$

In inequality (7), taking limit as $n \to \infty$ and using condition (9), we get,

$$\lim_{n\to\infty}\underline{\wp}_b(\underline{\varrho}_{n+k+1}, \underline{\varrho}_{n+k}) = 0.$$

This shows that $\langle \underline{\varrho}_n \rangle_{n\in N}$ is Cauchy and owing to completeness of the space $(\mho, \underline{\wp}_b, \underline{\Omega})$, converges to some point, say $\underline{\varsigma}$. Now, consider the inequality (8),

$$\underline{\wp}_b(\underline{\varrho}_{n+1}, \underline{\varrho}_n) \leq \underline{\xi}^n\underline{\wp}_b(\underline{\varrho}_1, \underline{\varrho}_0)$$

$$\underline{\wp}_b(\underline{I}_\vartheta\underline{\varrho}_n, \underline{\varrho}_n) \leq \underline{\xi}^n\underline{\wp}_b(\underline{\varrho}_1, \underline{\varrho}_0)$$

Now taking limit as $n \to \infty$, we get,

$$\frac{1}{s}\underline{\wp}_b(\underline{I}_\vartheta\underline{\varsigma}, \underline{\varsigma}) = 0$$

$$\underline{\wp}_b(\underline{I}_\vartheta\underline{\varsigma}, \underline{\varsigma}) = 0.$$

Thus, $\underline{I}_\vartheta\underline{\varsigma} = \underline{\varsigma}$, and therefore $\underline{\varsigma}$ is a fixed point of $\underline{I}_\vartheta$. But by using Lemma 2, we must have

$$Fix(\underline{I}) = Fix(\underline{I}_\vartheta),$$

and $Fix(\underline{I}) = \{\underline{\varrho}\}$, i.e. Fixed point of $\underline{I}$ is $\underline{\varrho}$, which is unique.

So, $\underline{\varsigma} = \underline{\varrho}$ and thus $\langle \underline{\varrho}_n \rangle_{n\in N}$ obtained from the above iteration converges to $\underline{\varrho}$.

3. Using inequalities (10) and (8), we have

$$\wp_b(\varrho_{n+m},\varrho_n) \le \frac{\xi}{1-\xi}[\wp_b(\varrho_{n+m-1},\varrho_{n+m})$$

$$+\wp_b(\varrho_n,\varrho_{n-1})]$$

$$\le \frac{\xi}{1-\xi}[\xi^{n+m-1}\wp_b(\varrho_1,\varrho_0)$$

$$+\xi^{n-1}\wp_b(\varrho_1,\varrho_0)]$$

$$= \frac{\xi^n(\xi^m+1)}{1-\xi}\wp_b(\varrho_1,\varrho_0)$$

Now letting $m \to \infty$, we get,

$$\frac{1}{s}\wp_b(\varrho,\varrho_n) \le \frac{\xi^n}{1-\xi}\wp_b(\varrho_1,\varrho_0) \qquad \ldots(11)$$

and $\wp_b(\varrho_{n+m},\varrho_n) \le \frac{\xi}{1-\xi}[\wp_b(\varrho_{n+m-1},\varrho_{n+m})$

$$+\wp_b(\varrho_n,\varrho_{n-1})]$$

$$\le \frac{\xi}{1-\xi}[\xi^{m-1}\wp_b(\varrho_{n-1},\varrho_n)$$

$$+\wp_b(\varrho_n,\varrho_{n-1})]$$

$$= \frac{\xi(\xi^{m-1}+1)}{1-\xi}\wp_b(\varrho_{n-1},\varrho_n)$$

Now letting $m \to \infty$, we get,

$$\frac{1}{s}\wp_b\left(\varrho,\varrho_n\right) \le \frac{\xi}{1-\xi}\wp_b\left(\varrho_{n-1},\varrho_n\right). \qquad \ldots(12)$$

After combining (11) and (12), the following error estimate holds

$$\frac{1}{s}\wp_b(\varrho_{n+i-1},\varrho) \le \frac{\xi^i}{1-\xi}\wp_b(\varrho_n,\varrho_{n-1}).$$

Hence, the result.

The following example illustrates importance of the above theorem.

Example 1: *Let the set of non-negative real numbers be* $\mho = R_0^+$ *and* $\wp_b(\varrho,\varsigma) = |\varrho-\varsigma|^3 + |\varrho-\varsigma|$ *for all* $\varrho,\varsigma \in \mho$. *Here, we perceive that*

1. $\wp_b(\varrho,\varsigma) \ge 0$ for all $\varrho,\varsigma \in \mho$;
2. $\wp_b(\varrho,\varsigma) = 0 \Leftarrow \varrho = \varsigma$;
3. $\wp_b(\varrho,\varsigma) = \wp_b(\varsigma,\varrho)$;
4. $\wp_b\left(\varrho,\varsigma\right) \le 4\left[\wp_b\left(\varrho,\eta\right) + \wp_b\left(\eta,\varsigma\right)\right]$, $\eta \in \mho$

   as $\wp_b\left(\varrho,\varsigma\right)$

   $= |\varrho-\varsigma|^3 + |\varrho-\varsigma|$

$$= \left|(\varrho-\eta)+(\eta-\varsigma)\right|^3 + |(\varrho-\eta)+(\eta-\varsigma)|$$

$$\le 4[|\varrho-\eta|)^3 + |\eta-\varsigma|^3)] + |\varrho-\eta| + |\eta-\varsigma|$$

$$\le 4[|\varrho-\eta|)^3 + |\varrho-\eta|] + 4[|\eta-\varsigma|^3) + |\eta-\varsigma|]$$

$$= 4[\wp_b(\varrho,\eta) + \wp_b(\eta,\varsigma)].$$

We define the convex structure $\Omega: \mho \times \mho \times \left\{\frac{1}{2}\right\} \to \mho$ as $\Omega(\varrho,\varsigma;\vartheta) = \frac{\varrho+\varsigma}{2}$, for any $\varrho,\varsigma \in \mho$ and $\vartheta \in [0,1]$. As a consequence,

$$\wp_b(\eta,\Omega(\varrho,\varsigma;\vartheta)) = \left|\eta - \frac{\varrho+\varsigma}{2}\right|^3 + \left|\eta - \frac{\varrho+\varsigma}{2}\right|$$

$$\le \left(\frac{1}{2}|\eta-\varrho| + \frac{1}{2}|\eta-\varsigma|\right)^3 + \left(\frac{1}{2}|\eta-\varrho| + \frac{1}{2}|\eta-\varsigma|\right)$$

$$\le 4\left[\frac{1}{8}|\eta-\varrho|^3 + \frac{1}{8}|\eta-\varsigma|^3\right] + \left(\frac{1}{2}|\eta-\varrho| + \frac{1}{2}|\eta-\varsigma|\right)$$

$$= \frac{1}{2}[|\eta-\varrho|^3 + |\eta-\varrho|] + \frac{1}{2}[|\eta-\varsigma|^3 + |\eta-\varsigma|]$$

$$= \vartheta[|\eta-\varrho|^3 + |\eta-\varrho|] + (1-\vartheta)[|\eta-\varsigma|^3$$

$$+|\eta-\varsigma|]$$

$$= \vartheta\wp_b(\eta,\varrho) + (1-\vartheta)\wp_b(\eta,\varsigma).$$

Thus, for $s \ge 4$, $(\mho,\wp_b,\Omega)$ is a convex b-metric space .

Let the mapping $I: \mho \to \mho$ be defined as

$$\underline{I}(\underline{\varrho}) = \{\frac{\underline{\varrho}}{17}, \underline{\varrho} \in \Lambda = [0,1)\ \frac{1}{20\underline{\varrho}}, \underline{\varrho} \in \Sigma = [1,\infty).$$

The following cases exist:

1. If both $\underline{\varrho}, \underline{\varsigma} \in \Lambda$, then the inequality (7) holds.
2. If $\underline{\varrho} \in \Lambda$ and $\underline{\varsigma} \in \Sigma$, then

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \left[|\underline{I\varrho} - \underline{I\varsigma}|^3 + |\underline{I\varrho} - \underline{I\varsigma}|\right] - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma})$$

$$+\underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \left[\left|\frac{\underline{\varrho}}{17} - \frac{1}{20\underline{\varsigma}}\right|^3 + \left|\frac{\underline{\varrho}}{17} - \frac{1}{20\underline{\varsigma}}\right|\right] - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma})$$

$$+\underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\left[\frac{1}{17^2}\left|\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right|^3 + \left|\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right|\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})].$$

If $\underline{\varrho} > \frac{17}{20\underline{\varsigma}}$, then

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\left[\frac{1}{17^2}\left(\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right)^3 + \left(\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$< \frac{1}{17}\left[\left(\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right)^3 + \left(\underline{\varrho} - \frac{17}{20\underline{\varsigma}}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$< \frac{1}{17}\left[\left(\underline{\varrho} - \frac{1}{20\underline{\varsigma}}\right)^3 + \left(\underline{\varrho} - \frac{1}{20\underline{\varsigma}}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= -\frac{1}{17}\underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho}) \leq 0.$$

If $\underline{\varrho} < \frac{17}{20\underline{\varsigma}}$, then

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\left[\frac{1}{17^2}\left(\frac{17}{20\underline{\varsigma}} - \underline{\varrho}\right)^3 + \left(\frac{17}{20\underline{\varsigma}} - \underline{\varrho}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\left[\frac{1}{17^2}\left(\frac{1}{\underline{\varsigma}} - \underline{\varrho}\right)^3 + \left(\frac{1}{\underline{\varsigma}} - \underline{\varrho}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$< \frac{1}{17}\left[\frac{1}{17^2}\left(\underline{\varsigma} - \underline{\varrho}\right)^3 + \left(\underline{\varsigma} - \underline{\varrho}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$< \frac{1}{17}\left[\left(\underline{\varsigma} - \frac{\underline{\varrho}}{17}\right)^3 + \left(\underline{\varsigma} - \frac{\underline{\varrho}}{17}\right)\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \frac{1}{17}\underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= -\frac{1}{17}\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) \leq 0.$$

Thus, we have

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) \leq \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})] \quad \ldots(13)$$

3. If $\underline{\varrho} \in \Sigma$ and $\underline{\varsigma} \in \Lambda$,

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) \leq \frac{1}{17}\left[\underline{\wp}_b\left(\underline{\varrho}, \underline{I\varsigma}\right) + \underline{\wp}_b\left(\underline{\varsigma}, \underline{I\varrho}\right)\right] \quad \ldots(14)$$

4. If both $\underline{\varrho}, \underline{\varsigma} \in \Sigma = [1, \infty)$

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) - \frac{1}{17}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$= \left[\left|\frac{1}{20\underline{\varrho}} - \frac{1}{20\underline{\varsigma}}\right|^3 + \left|\frac{1}{20\underline{\varrho}} - \frac{1}{20\underline{\varsigma}}\right|\right]$$

$$-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$=\frac{1}{20}\left[\frac{1}{20^2}\left|\frac{1}{\underline{\varrho}}-\frac{1}{\underline{\varsigma}}\right|^3+\left|\frac{1}{\underline{\varrho}}-\frac{1}{\underline{\varsigma}}\right|\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})].$$

If $\underline{\varsigma}>\underline{\varrho}$, then

$$\underline{\wp}_b(\underline{I\varrho},\underline{I\varsigma})-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$\leq\frac{1}{20}\left[\frac{1}{20^2}\left(\underline{\varrho}-\frac{1}{\underline{\varsigma}}\right)^3+\left(\underline{\varrho}-\frac{1}{\underline{\varsigma}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{20}\left[\frac{1}{20^2}\left(\underline{\varrho}-\frac{1}{20\underline{\varsigma}}\right)^3+\left(\underline{\varrho}-\frac{1}{20\underline{\varsigma}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{20}\left[\left(\underline{\varrho}-\frac{1}{20\underline{\varsigma}}\right)^3+\left(\underline{\varrho}-\frac{1}{20\underline{\varsigma}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$\leq\frac{1}{20}\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{17}\underline{\wp}_b\left(\underline{\varrho},\underline{I\varsigma}\right)-\frac{1}{17}\left[\underline{\wp}_b\left(\underline{\varrho},\underline{I\varsigma}\right)+\underline{\wp}_b\left(\underline{\varsigma},\underline{I\varrho}\right)\right]$$

$$=-\frac{1}{17}\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})\leq 0.$$

If $\underline{\varrho}>\underline{\varsigma}$, then

$$\underline{\wp}_b(\underline{I\varrho},\underline{I\varsigma})-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$\leq\frac{1}{20}\left[\frac{1}{20^2}\left(\underline{\varsigma}-\frac{1}{\underline{\varrho}}\right)^3+\left(\underline{\varsigma}-\frac{1}{\underline{\varrho}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{20}\left[\frac{1}{20^2}\left(\underline{\varsigma}-\frac{1}{20\underline{\varrho}}\right)^3+\left(\underline{\varsigma}-\frac{1}{20\underline{\varrho}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{20}\left[\left(\underline{\varsigma}-\frac{1}{20\underline{\varrho}}\right)^3+\left(\underline{\varsigma}-\frac{1}{20\underline{\varrho}}\right)\right]-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$\leq\frac{1}{20}\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$<\frac{1}{17}\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})-\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})]$$

$$=-\frac{1}{17}\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})\leq 0,$$

which infers that for all $\underline{\varrho},\underline{\varsigma}\in\mho$

$$\underline{\wp}_b(\underline{I\varrho},\underline{I\varsigma})\leq\frac{1}{17}[\underline{\wp}_b(\underline{\varrho},\underline{I\varsigma})+\underline{\wp}_b(\underline{\varsigma},\underline{I\varrho})].$$

Therefore, for $k=\frac{1}{17}<\frac{1}{s^2}$, $\underline{I}$ satisfies the inequality.

Generate the sequence iteration
$\underline{\varrho}_n=\underline{\Omega}(\underline{\varrho}_{n-1},\underline{I\varrho}_{n-1};\vartheta)$ with $0<\vartheta=\frac{1}{2}<1$.

There are two possibilities for $\underline{\varrho}_0$:

1. If $\underline{\varrho}_0\in\Lambda$, then

$$\underline{I\varrho}_0=\frac{\underline{\varrho}_0}{17}$$

$$\underline{\varrho}_1=\frac{1}{2}\underline{\varrho}_0+\frac{1}{2}T\underline{\varrho}_0=\left(\frac{9}{17}\right)\underline{\varrho}_0$$

$$\underline{\varrho}_2=\frac{1}{2}\underline{\varrho}_1+\frac{1}{2}T\underline{\varrho}_1=\left(\frac{9}{17}\right)^2\underline{\varrho}_0$$

$$\vdots$$

$$\underline{\varrho}_n=\frac{1}{2}\underline{\varrho}_{n-1}+\frac{1}{2}T\underline{\varrho}_{n-1}=\left(\frac{9}{17}\right)^n\underline{\varrho}_0.$$

Certainly, $\underline{\varrho}_n\to 0$ as $n\to\infty$.

2. If $\underline{\varrho}_0\in\Sigma$, then

$$\underline{I\varrho}_0=\frac{1}{20\underline{\varrho}_0}$$

$$\underline{\varrho}_1=\frac{1}{2}\underline{\varrho}_0+\frac{1}{2}\underline{I\varrho}_0=\frac{1}{2}\underline{\varrho}_0+\frac{1}{2}\cdot\frac{1}{20\underline{\varrho}_0}$$

$$\frac{\underline{\varrho}_1}{\underline{\varrho}_0}=\frac{1}{2}+\frac{1}{40}\cdot\frac{1}{\underline{\varrho}_0^2}\leq\frac{21}{40}.$$

If $\underline{\varrho}_1 \in \Lambda$, as $n \to \infty, \underline{\varrho}_n \to 0$ as in the previous case. If $\underline{\varrho}_1 \in \Sigma$, then $\frac{\underline{\varrho}_2}{\underline{\varrho}_1} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\underline{\varrho}_1^2} = \frac{21}{40}$. Continuing in comparable manner, we presume that $\underline{\varrho}_{n-1} \in \Sigma$ yielding

$$\frac{\underline{\varrho}_n}{\underline{\varrho}_{n-1}} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\underline{\varrho}_{n-1}^2} = \frac{21}{40}$$

and

$$\frac{\underline{\varrho}_n}{\underline{\varrho}_0} = \frac{\underline{\varrho}_1}{\underline{\varrho}_0} \cdot \frac{\underline{\varrho}_2}{\underline{\varrho}_1} \cdots \frac{\underline{\varrho}_n}{\underline{\varrho}_{n-1}} = \left(\frac{21}{40}\right)^n,$$

and hence $lim_{n\to\infty}\underline{\varrho}_n = 0$.

Now, if $\underline{\varrho}_0 \in \Lambda$, consider

$$\underline{\wp}_b(\underline{\varrho}_n, \underline{I\varrho}_n) = |\underline{\varrho}_n - \underline{I\varrho}_n|^3 + |\underline{\varrho}_n - \underline{I\varrho}_n|$$

$$= \left|\left(\frac{9}{17}\right)^n \underline{\varrho}_0 - \left(\frac{9}{17}\right)^{n+1} \underline{\varrho}_0\right|^3$$

$$+ \left|\left(\frac{9}{17}\right)^n \underline{\varrho}_0 - \left(\frac{9}{17}\right)^{n+1} \underline{\varrho}_0\right|$$

$$= \left(\frac{9}{17}\right)^{3n} \left(\frac{8}{17}\right)^3 \underline{\varrho}_0^3 + \left(\frac{9}{17}\right)^n \left(\frac{8}{17}\right) \underline{\varrho}_0.$$

Clearly,

$$\lim_{n\to\infty} \underline{\wp}_b(\underline{\varrho}_n, \underline{I\varrho}_n) = 0. (15)$$

Also, if $\underline{\varrho}_0 \in \Sigma$, then

$$\underline{\wp}_b\left(\underline{\varrho}_n, \underline{I\varrho}_n\right)$$

$$= |\underline{\varrho}_n - \underline{I\varrho}_n|^3 + |\underline{\varrho}_n - \underline{I\varrho}_n| \ \le |\underline{\varrho}_n|^3 + |\underline{\varrho}_n|$$

$$= \left|\left(\frac{21}{40}\right)^n \underline{\varrho}_0\right|^3 + \left|\left(\frac{21}{40}\right)^n \underline{\varrho}_0\right|$$

$$= \left(\frac{21}{40}\right)^{3n} \underline{\varrho}_0^3 + \left(\frac{21}{40}\right)^n \underline{\varrho}_0.$$

Clearly,

$$\lim_{n\to\infty} \underline{\wp}_b(\underline{\varrho}_n, \underline{I\varrho}_n) = 0. \qquad \ldots(16)$$

Thus, from equations 15 and 16, we get

$$inf\{\underline{\wp}_b(\underline{\varrho}, \underline{I\varrho});\ \underline{\varrho} \in \mho\} = 0.$$

Remark 1: *If* $\underline{\varrho} = 0$ *and* $\underline{\varsigma} = \frac{1}{2}$ *, then* $\underline{I\varrho} = 0$ *and* $\underline{I\varsigma} = \frac{1}{20}$ *resulting in*

$$\underline{\wp}_b(\underline{I\varrho}, \underline{I\varsigma}) = \underline{\kappa}[\underline{\wp}_b(\underline{\varrho}, \underline{I\varsigma}) + \underline{\wp}_b(\underline{\varsigma}, \underline{I\varrho})]$$

$$\left(\frac{1}{20}\right)^3 + \left(\frac{1}{20}\right) \le \underline{\kappa}\left[\left(\frac{1}{20}\right)^3 + \left(\frac{1}{20}\right) + 1^3 + 1\right] \quad \ldots(17)$$

which is true for all $\underline{\kappa} \ge \frac{1}{41} < \frac{1}{s(s+1)}$ and $\underline{\kappa} \ge \frac{1}{41} > \frac{1}{s^2(s^2+1)}$ and therefore Theorem 3 is an extension of Chatterjea fixed point theorem proved by Rathee et al.[11]

APPLICATION TO CAUCHY PROBLEM

Consider space $\mho = C[\underline{\delta}, \underline{\xi}] = \{\underline{\varrho}(\varpi);\ \underline{\varrho}: [\underline{\delta}, \underline{\xi}] \to R\}$ and the Cauchy problem

$$\frac{d\underline{\varsigma}(\varpi)}{d\varpi} = \phi\left(\varpi, \underline{\varsigma}(\varpi)\right), with\ \underline{\varsigma}(\varpi_0) = \underline{\varsigma}_0, \} \quad \ldots(18)$$

where $\phi(\varpi, \underline{\varsigma}(\varpi)): [\underline{\delta}, \underline{\xi}] \times R \to R$, $\underline{\varsigma}(\varpi)$ are continuous functions in $[\underline{\delta}, \underline{\xi}]$ and $\underline{\varsigma}(\varpi)$ is differentiable in $[\underline{\delta}, \underline{\xi}]$. Here, ϖ_0 is a point in the interior of the interval $[\underline{\delta}, \underline{\xi}]$.

This Cauchy Problem 18 is identical to the following integral equation:

$$\underline{\varsigma}(\varpi) = \underline{\varsigma}_0 + \int_{\varpi_0}^{\varpi} \phi(\underline{\Lambda}, \underline{\varsigma}(\underline{\Lambda}))\, d\underline{\Lambda}. \qquad \ldots(19)$$

Define $\underline{\wp}_b: \mho \times \mho \to [0, \infty)$ by

$$\underline{\wp}_b(\underline{\varrho}, \underline{\varsigma}) = sup_{\varpi\in[\underline{\delta},\underline{\xi}]}(\underline{\varrho}(\varpi) - \underline{\varsigma}(\varpi))^2\ \forall\ \underline{\varrho}, \underline{\varsigma} \in \mho$$

Define $\underline{\Omega}: \mho \times \mho \times [0,1] \to \mho$ as

$$\underline{\Omega}(\underline{\varrho}, \underline{\varsigma}; \vartheta) = \vartheta\underline{\varrho} + (1 - \vartheta)\underline{\varsigma}.$$

Additionally, consider a self-mapping $\underline{I}: \mho \to \mho$ defined as

$$\underline{I\varsigma}(\varpi) = \underline{\varsigma}_0 + \int_{\varpi_0}^{\varpi} \Psi(\underline{\Lambda}, \underline{\varsigma}(\underline{\Lambda}))\, d\underline{\Lambda}\ \forall\ \varpi, \underline{\Lambda} \in [\underline{\delta}, \underline{\xi}].$$

Then, existence of unique fixed point of mapping $\underline{I}$ implies the existence and uniqueness of solution of the integral equation 19 and hence, the Cauchy Problem 18.

Lemma 3: *Let* $\mho = C[\underline{\delta}, \underline{\xi}] = \{\underline{\varrho}(\varpi);\ \underline{\varrho}: [\underline{\delta}, \underline{\xi}] \to R\}$ *and define* $\underline{\wp_b}: \mho \times \mho \to [0, \infty)$ *by*

$$\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) = sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\varrho}(\varpi) - \underline{\varsigma}(\varpi))^2\ \forall\ \underline{\varrho}, \underline{\varsigma} \in \mho$$

Define the convex structure $\underline{\Omega}: \mho \times \mho \times [0,1] \to \mho$ as

$$\underline{\Omega}(\underline{\varrho}, \underline{\varsigma}; \vartheta) = \vartheta\underline{\varrho} + (1-\vartheta)\underline{\varsigma}\ \forall\ \underline{\varrho}, \underline{\varsigma} \in \mho.$$

Then, $(\mho, \underline{\wp_b}, \underline{\Omega})$ is a convex b-metric space.

Proof. We observe that

1. $\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) \geq 0\ \forall\ \underline{\varrho}, \underline{\varsigma} \in \mho.$
2. $\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) = 0 \Leftarrow \underline{\varrho} = \underline{\varsigma}.$
3. $\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) = \underline{\wp_b}(\underline{\varsigma}, \underline{\varrho}).$
4. $\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) \leq 2[\underline{\wp_b}(\underline{\varrho}, \underline{\eta}) + \underline{\wp_b}(\underline{\eta}, \underline{\varsigma})]$ as

$$\underline{\wp_b}(\underline{\varrho}, \underline{\varsigma}) = sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\varrho}(\varpi) - \underline{\varsigma}(\varpi))^2$$
$$= sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\varrho}(\varpi) - \underline{\eta}(\varpi) + \underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2$$
$$\leq 2\{sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\varrho}(\varpi) - \underline{\eta}(\varpi))^2$$
$$+ sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2\}$$
$$= 2[\underline{\wp_b}(\underline{\varrho}, \underline{\eta}) + \underline{\wp_b}(\underline{\eta}, \underline{\varsigma})].$$

Also, for $\underline{\Omega}\left(\underline{\varrho}, \underline{\varsigma}; \vartheta\right) = \vartheta\underline{\varrho} + (1-\vartheta)\underline{\varsigma}\ \forall\ \underline{\varrho}, \underline{\varsigma} \in \mho$, we have

$$\underline{\wp_b}\left(\underline{\eta}, \underline{\Omega}\left(\underline{\varrho}, \underline{\varsigma}; \vartheta\right)\right)$$
$$= sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\eta}(\varpi) - \underline{\Omega}(\underline{\varrho}(\varpi), \underline{\varsigma}(\varpi); \vartheta))^2$$
$$= sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\eta}(\varpi) - \{\vartheta\underline{\varrho}(\varpi) + (1-\vartheta)\underline{\varsigma}(\varpi)\})^2$$
$$\leq sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\vartheta|\underline{\eta}(\varpi) - \underline{\varrho}(\varpi)| + (1-\vartheta)|\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi)|)^2$$
$$= sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}[\vartheta^2(\underline{\eta}(\varpi) - \underline{\varrho}(\varpi))^2$$
$$+ (1-\vartheta)^2(\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2 + 2\vartheta(1-\vartheta)|\underline{\eta}(\varpi) - \underline{\varrho}(\varpi)||\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi)|]$$
$$\leq sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}[\vartheta^2(\underline{\eta}(\varpi) - \underline{\varrho}(\varpi))^2$$
$$+ (1-\vartheta)^2(\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2 + \vartheta(1-\vartheta)\{(\underline{\eta}(\varpi) - \underline{\varrho}(\varpi))^2 + (\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2\}]$$
$$\leq \vartheta sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\eta}(\varpi) - \underline{\varrho}(\varpi))^2$$
$$+ (1-\vartheta) sup_{\varpi \in [\underline{\delta}, \underline{\xi}]}(\underline{\eta}(\varpi) - \underline{\varsigma}(\varpi))^2$$
$$= \vartheta\underline{\wp_b}(\underline{\eta}, \underline{\varrho}) + (1-\vartheta)\underline{\wp_b}(\underline{\eta}, \underline{\varsigma})$$

Thus, for $s \geq 2$, $(\mho, \underline{\wp_b}, \underline{\Omega})$ is convex b-metric space.

Theorem 4: *Consider*

$$\left|\phi(\underline{\Lambda}, \underline{\varrho}(\underline{\Lambda})) - \phi(\underline{\Lambda}, \underline{\varsigma}(\underline{\Lambda}))\right| \leq \left[\underline{\kappa}M(\underline{\varrho}, \underline{\varsigma})\right]^{\frac{1}{2}}$$

for all $\varpi, \underline{\Lambda} \in [\underline{\delta}, \underline{\xi}]$; $\underline{\varrho}, \underline{\varsigma} \in \mho$ and some $\underline{\kappa} < \frac{1}{(\underline{\xi} - \underline{\delta})^2} \leq \frac{1}{s^2}$

where

$$M(\underline{\varrho}, \underline{\varsigma}) = \underline{\wp_b}(\underline{\varrho}, \underline{I}\underline{\varsigma}) + \underline{\wp_b}(\underline{\varsigma}, \underline{I}\underline{\varrho})\}.$$

Then, a unique solution exists for Integral Equation 19.

Proof. Consider

$$(\underline{I}\underline{\varrho}(\varpi) - \underline{I}\underline{\varsigma}(\varpi))^2$$
$$\leq \left(\int_{\varpi_0}^{\varpi}\left|\phi(\underline{\Lambda}, \underline{\varrho}(\underline{\Lambda}) - \phi(\underline{\Lambda}, \underline{\varsigma}(\underline{\Lambda}))\right|\, d\underline{\Lambda}\right)^2$$
$$\leq \left(\int_{\varpi_0}^{\varpi}\left[\underline{\kappa}M(\underline{\varrho}, \underline{\varsigma})\right]^{\frac{1}{2}}\, d\underline{\Lambda}\right)^2$$
$$\leq \underline{\kappa}M(\underline{\varrho}, \underline{\varsigma})\left(\int_{\varpi_0}^{\varpi} d\underline{\Lambda}\right)^2$$
$$= \underline{\kappa}M(\underline{\varrho}, \underline{\varsigma})(\varpi - \varpi_0)^2$$
$$\leq \underline{\kappa}M(\underline{\varrho}, \underline{\varsigma})(\underline{\xi} - \underline{\delta})^2$$

implying Integral Equation 19 and hence, the Cauchy Problem 18, has a unique solution that is unique as all the hypothesis of Theorem 3 are satisfied.

CONCLUSION

As an extension of the elementary result of Rathee et al.[11], we put forward a fixed point theorem that ensures the existence of a fixed point for Chatterjea contraction in the setting of convex b-metric spaces.

The Kransnoselkij iterative process is used for approximating the fixed point and the conclusions drawn here and use these conclusions to solve Cauchy problem.

OPEN PROBLEM

Rathee et al.[11] ensured the existence of fixed point for Chatterjea contraction for the constant $\underline{\kappa} \in [0, \frac{1}{s^2(s^2+1)})$. In addition, we extended the same for $\underline{\kappa} \in [\frac{1}{s^2(s^2+1)}, \frac{1}{s(s+1)})$. Is it viable to further relax the condition for $\underline{\kappa} \in [\frac{1}{s(s+1)}, \frac{1}{2})$?

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